change amounts to throwing the first factor in (12) into the denominator. In place of (13) and (14) we get

$$L(La+2M)=0,\ L^2(2-b)+M^2=4A.$$

According as the group is $C_4$ or $G_4$, we have

$$L(a-v)+M=\mp(Lv+M).$$

Hence for $C_4$, $La+2M=0$, $L^2B=16A$, so that $L$ and $M$ are rational if and only if $\sqrt{(AB)}$ is rational. For $G_4$, we have $L=0$, $M^2=4A$, so that $\sqrt{A}$ must be rational. In either case the expression for $W(x_1)$ has $x_1-1/x_1$ in the denominator, but is much shorter than that in the paper.

# ON CERTAIN TRANSCENDENTAL FUNCTIONS DEFINED BY A SYMBOLIC EQUATION.*

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## 1. Introduction and Definition.

The object of this paper is to define a certain large class of transcendental functions of real variables to which the previous consideration of a geometric subject† gave rise and to point out a few of their general properties. Attention will be confined chiefly to questions of continuity and development in series.

We shall take

(1) $$u=f(x) \text{ and } v=f(y),$$

and put

(2) $$u^v=v^u;$$

thus we have an equation which is satisfied identically when $x$ is put for $y$; and therefore one solution of the symbolic equation (2) is $y=x$, whatever function is denoted by $f$. We discard at once this solution as not pertinent to the present discussion.

In order to obtain a second solution, put

(3) $$v=\mu u,$$

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†Carmichael, The American Mathematical Monthly, Vol. XIII, pp. 221-226, 1906.

where $\mu$ is another variable whose law of variation is determined conjointly by equations (2) and (3). A substitution of the latter value of $v$ in the former equation will yield, by an easy reduction,

(4) $$u=f(x)=\mu^{1/(\mu-1)},$$
(5) $$v=f(y)=\mu^{\mu/(\mu-1)}.$$

Now suppose that the functional equations (4) and (5) are solved, giving

(6) $$x=g(\mu^{1/(\mu-1)}),$$
(7) $$y=g(\mu^{\mu/(\mu-1)}),$$

where $g$ is the function inverse to $f$. Equations (6) and (7) define $y$ as a function of $x$; this is the general function to be studied.*

It should be noticed that the preceding discussion does not show that the two given solutions of (2) are the only ones possible. On the other hand it seems indeed quite probable that such a transcendental equation may have other sets of real solutions; and these may give rise to other quite as interesting functions as those of this paper.

## 2. Continuities and Discontinuities of the Expressions.

In the present section we shall find the conditions under which $\mu^{1/(\mu-1)}$ and $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$ are continuous. It is evident that they are both continuous only when $\mu$ itself is continuous.

We shall find that $\mu^{1/(\mu-1)}$ has a continuous graph in the first quadrant. It will now be shown that its graph is discontinuous in the second, third and fourth quadrants; and the nature of the discontinuity will be exhibited.

In the second and third quadrants $\mu$ is negative. Consider the value $\mu=-p/q$, where $p/q$ is a fraction in its lowest terms and $p$ and $q$ are both positive. At most one of them can be even. Then

(8) $$u=\mu^{1/(\mu-1)}=\left(-\frac{p}{q}\right)^{\frac{-q}{p+q}}=\left(-\frac{q}{p}\right)^{\frac{q}{p+q}}.$$

Now $u$ is imaginary if both $p$ and $q$ are odd; for then it is equal to an even root of a negative number. And $u$ is real if one of the numbers $p$ and $q$ is odd and the other even. Now consider the five values of $\mu$:

$$-\frac{(2n+1)\,p-2}{(2n+1)q},\quad -\frac{(2n+1)\,p-1}{(2n+1)q},\quad -\frac{p}{q},\quad -\frac{(2n+1)\,p+1}{(2n+1)q},\quad -\frac{(2n+1)\,p+2}{(2n+1)q},$$

*The curve $u^v=v^u$ was studied by Dan Bernoulli and by Euler. The latter gave a parametric representation similar to that here employed, and plotted the curve in the first quadrant. (See Euler, *Introductio*, Vol. 2, p. 294, edition of 1748.) A study from the viewpoint of function theory is thought to be of sufficient importance to justify this presentation. The function well illustrates several important conceptions of the general theory.

where $n$ is a positive integer and one of the numbers $p$, $q$ is odd and the other even. If $p$ is even, then in the second and fourth fractions, both numerator and denominator are odd, and $u$ is imaginary for each of these values; but to the first and fifth values there evidently correspond real points of the locus. By taking $n$ large at pleasure the first, second, fourth, and fifth values of $\mu$ can be brought indefinitely near its third value; and hence on each side of the point corresponding to $\mu = -p/q$ is a point infinitely near to it in the domain of $\mu$, such that to this point corresponds no point on the locus. Moreover this point is seen to lie between two others infinitely near each other such that to each of these corresponds a point of the locus.

If $q$ is even we may consider the set of values

$$-\frac{(2n+1)p}{(2n+1)q+2},\quad -\frac{(2n+1)p}{(2n+1)q+1},\quad -\frac{p}{q},\quad -\frac{(2n+1)p}{(2n+1)q-1},\quad -\frac{(2n+1)p}{(2n+1)q-2}$$

in which $p$ is necessarily odd. A discussion will yield the same conclusion as that obtained when $p$ is even.

These discussions lead readily to the following theorems:

*When $\mu$ is negative there is in every interval, however small, an infinite number of points on the graph while at the same time there are in the same interval in infinite number of discontinuities.* Further, it is clear that *in any such interval $\mu^{1/(\mu-1)}$ is dense, while the points of discontinuity also constitute a dense aggregate.* Finally, *for each point of discontinuity arising as above the function $\mu^{1/(\mu-1)}$ is imaginary.*

In the fourth quadrant $u$ is negative while $\mu$ is positive. A discussion similar to that above will lead to the conclusion that *in the fourth quadrant the graph of $\mu^{1/(\mu-1)}$ is infinitely discontinuous.* The discontinuity here arises not by $u$ taking on imaginary values, but by its taking (when real) only positive values for values of $\mu$ infinitely near to any value of $\mu$ which corresponds to a real point in the fourth quadrant. This is readily seen from a discussion of the following five values of $\mu$:

$$\frac{(2n+1)p-2}{(2n+1)q},\quad \frac{(2n+1)p-1}{(2n+1)q},\quad \frac{p}{q},\quad \frac{(2n+1)p+1}{(2n+1)q},\quad \frac{(2n+1)p+2}{(2n+1)q}$$

The reader can now readily supply the discussion requisite to establish the fact of infinite discontinuity in the present instance.

The different ways in which the discontinuity arises in these two cases is interesting. A more detailed discussion would show that in the third quadrant the discontinuity arises from the introduction both of imaginary values and of such as these just referred to in the discussion of the locus in the fourth quadrant. It is thence easy to show that along the same branch of the curve in the third quadrant there are now three dense aggre-

gates: (1) the aggregate of real points of the locus; (2) the aggregate of points of discontinuity corresponding to those values of $\mu$ for which $\mu^{1/(\mu-1)}$ becomes imaginary; (3) the aggregate of points of discontinuity corresponding to those values of $\mu$ for each of which $\mu^{1/(\mu-1)}$ takes on a real positive value but no negative value.

From the preceding discussion it follows that if the graph of $\mu^{1/(\mu-1)}$ possesses a continuous branch it lies in the first quadrant. We proceed to establish the fact of its continuity here.

Let $a$ be any value of $\mu$ except $\mu=1$, and let the vicinity of $a$ be the interval $a-h$ to $a+h$. Represent $\mu^{1/(\mu-1)}$ by $F(\mu)$. Then the well known necessary and sufficient condition of continuity that for each $e>0$ there must exist an $h$ such that

$$|\ F(a+h)-F(a-h)\ |<e$$

is evidently fulfilled. Hence $F(\mu)$ is continuous everywhere except possibly at $\mu=1$. But as $\mu \doteq 1$ from either side,

$$\mu^{1/(\mu-1)}=[1+(\mu-1)]^{1/(\mu-1)} \doteq e=2.712..,;$$

and therefore it follows that the function is continuous at $\mu=1$. Hence it is everywhere continuous. That is, its locus has a continuous branch. As we have seen, this must be in the first quadrant.

Since $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$, the question of the continuity of this function does not require separate consideration; for, since $\mu$ is to be taken continuous, it is evident that $\mu^{\mu/(\mu-1)}$ is continuous for all values of $\mu$ for which $\mu^{1/(\mu-1)}$ is continuous.

Since $u=\mu^{1/(\mu-1)}$ and $v=\mu^{\mu/(\mu-1)}$, the preceding discussion leads readily to the conclusion that $v$ is a continuous function of $u$ at every point for which both $u$ and $v$ are positive, and equation (2) is satisfied. Likewise $u$ is a continuous function of $v$ under the same limitations. We may then have the theorem:

*$u$ and $v$ each is a continuous function of the other throughout the domain of positive rational and irrational numbers.*

Now, if in equations (6) and (7) the function denoted by $g$ is continuous, the preceding discussion yields readily the following:

*$y$ is a continuous function of $x$ throughout the domain represented by $\mu^{1/(\mu-1)}$ when the domain of $\mu$ is all positive rational or irrational numbers.*

## 3. Development in Series.

Resuming equation (4),

$$f(x)=u=\mu^{1/(\mu-1)}, \tag{9}$$

we shall first express $\mu$ as an infinite series in terms of $u$. This will be carried out by the aid of Lagrange's formula and through the help of certain simple substitutions, as follows:*

From (9) we have $u^{\mu-1}=\mu$. Let $u=e^z$. Then $e^{z(\mu-1)}=\mu$. Let $z(\mu-1)=t$; then $e^t=\mu=(\mu-1)+1=t/z+1$. Hence,

$$t=-z+ze^t. \tag{10}$$

If $t=a+b\,\phi(t)$, we have by Lagrange's formula,

$$t=a+b\,\phi(a)+\frac{b^2}{2\,!}\frac{d}{da}[\phi(a)]^2+\ldots+\frac{b^n}{n\,!}\left(\frac{d}{d\,a}\right)^{n-1}[\phi(a)]^n+\ldots \tag{11}$$

Here we have by comparison with (10), $\phi(t)=e^t$, $a=-z$, $b=z$; by a substitution of these values equation (11) reduces to

$$t=-z+ze^{-z}+z^2e^{-2z}+\frac{3z^3}{2!}e^{-3z}+\ldots+\frac{n^{n-2}z^n}{(n-1)\,!}e^{-nz}+\ldots$$

But $\mu=t/z+1$, as we have seen; and therefore equation (9) readily yields

$$\mu=e^{-z}+ze^{-2z}+\frac{3z^2}{2!}e^{-3z}+\ldots+\frac{n^{n-2}z^{n-1}}{(n-1)\,!}e^{-nz}+\ldots$$

Since $z=\log_e u$ and $e^{-z}=1/u$, this becomes

$$\mu=\frac{1}{u}\left[1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots\right],$$

and since $\mu\,u=v$, we have

$$v=1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots, \tag{12}$$

a development of $v$ in terms of $u$. Now since $u$ and $v$ enter equation (2) in just the same way, it is clear that the development of $u$ in terms of $v$ may be obtained simply by an interchange of $u$ and $v$ in (12); in other words, the series in (12) has the interesting property that it reverts into the same series in the other variable. Hence, if we denote this function by $T$, we have

*Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. XIII, pp. 18, 72; 1906.

$$u=T(v),\ v=T(u).$$

Therefore, $$v=T[T(v)]=T^2(v).$$

But $$v=T^{-1}[T(v)],$$

whatever function is denoted by $T$. Hence the above function $T$ is equal to its inverse function. Therefore we may think of $T$ as a functional operation such that

$$T^2=1, \text{ but } T\neq+1 \text{ or } -1.$$

We consider now the question of the convergency of the series in (12). We shall apply the test of the $(n+1)$th term divided by the $n$th term. To find the limiting value of this quotient, we have the following equations:

$$\frac{(n+1)^{n-1}}{n!}\left(\frac{\log_e u}{u}\right)^n \div \frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}$$

$$=\left(\frac{n+1}{n}\right)^{n-1}\frac{\log_e u}{u}=\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n\frac{\log_e u}{u}.$$

But $\underset{n\doteq\infty}{\text{limit}}\ \frac{n}{n+1}\left(1+\frac{1}{n}\right)^n=e$. Hence the above ratio is less than 1 and the series is convergent for every case for which $\frac{\log_e u}{u}<\frac{1}{e}$. Now it is easy to show that $\frac{\log_e u}{u}$ takes on its maximum value when $u=e$ and that this value is $1/e$. Hence the series in (12) is certainly convergent for every value of $u$ except $u=e$; and for this latter value the preceding discussion gives no answer. For this case, however, it may be shown by other means that $v=e$; for $u=\mu^{1/(\mu-1)}$ takes on the value $e$ only when $\mu=1$, $\mu$ being confined to real values; and since $v=\mu u$, $v=u$ for this particular value.

We append here also an interesting expansion of $v$ in exponential form in terms of $u$. From $u=\mu^{1/(\mu-1)}$ we have $u^{\mu-1}=\mu$; $u^\mu=\mu u$. Whence

$$\mu=\frac{u^\mu}{u}=\frac{u^{\frac{u^\mu}{u}}}{u}$$

Hence $\mu u=v=u^{\frac{u^{\frac{u^{\frac{u\cdots}{u}}}{u}}}{u}}$

Evidently the corresponding expansion of $u$ in terms of $v$ is gotten by an interchange of these two variables in the last equation.